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Equilibrium-range spectra of sand waves formed by flowing water

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Based on a dimensional consideration, the '-3 power law' on the spatial spectrum of sand waves formed by flowing water $S_{\eta\eta}(k)$ is derived for a large wave-number equilibrium subrange,

$$S_{\eta\eta}(k) \sim \alpha k^{-3},$$

where α is a constant depending on the angle of repose of sand particles and k denotes the wave-number.

Likewise, the frequency spectrum is shown to have the '-3 power law' range for higher frequencies as well as the '-2 power law' range for frequencies near a spectral peak.

These spectra are shown to agree with experimental data from various sources.

1. Introduction

Mechanism of fluvial sand movements has for many years been attracting the interest of not only hydraulicians, but also physicists, and many papers have been written on the problems such as the initiation of movement of sand particles, the suspension and transportation of sand and the formation of sand waves. Nevertheless, much remains to be solved.

A recent problem which is arousing controversy is statistical properties of sand waves formed by flowing water. From experimental investigation, it became evident that the sand waves are composed of a wide range of frequency components, contrary to the seemingly rather regular pattern, and in higher frequency or wave-number ranges the sand-wave spectra plotted on log-log scale decrease almost linearly with increasing frequencies or wave-numbers. Nordin & Algert (1966) proposed a Markov second-order linear model which was further developed by Ashida & Tanaka (1967), while the writer (Hino 1968) has derived preliminarily the '-3 power law' of sand-wave spectrum.

Generally speaking, the dimensional consideration is an efficient tool for deriving spectral characteristics. One of the best known is Kolmogorov's $-\frac{5}{3}$ power law (Batchelor 1953) for the inertial subrange of turbulent velocity fluctuations. In the inertial subrange where energy transfer is predominantly governed by the rate of energy dissipation ϵ as well as by the wave-number k, the molecular viscosity ν contributing nothing, the spectrum of velocity fluctuations is expressed by

$$E(k) = a\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{1}$$

where a is a universal constant named Kolmogorov constant.

Phillips's '-5 power law' on the wind-wave spectrum in the equilibrium range (Phillips 1958) is also familiar to us. In a well-developed sea generated by the wind, there is an equilibrium range of large wave-numbers (or high frequencies) in the spectrum, determined by the condition that the downward acceleration should not exceed the gravitational acceleration (g). Therefore, the spectrum is determined simply by g and angular frequencies (in radian/unit time) ω or g and k.

$$\Phi(\omega) \sim bg^2\omega^{-5}, \tag{2}$$

where b is a constant, and

$$\Psi(\mathbf{K}) \sim f(\theta)k^{-4}, \tag{3}$$

where θ is an angle specifying the direction of the vector wave-number $\mathbf{K}$ and $f(\theta)$ is a certain function of θ.

In this paper, extending further the previous discussion, the writer shows that there exists an equilibrium subrange in the sand-wave spectrum. On a dimensional ground, the '-3 power law' of the equilibrium spectrum is derived for large wave-numbers (or high frequencies). The result is compared with the experimental data.

2. Dimensional analysis

Wave-number spectrum

In the first place, the instantaneous spatial characteristics of sand waves are discussed.

Let η denote an elevation of sand waves from the mean level of sand bed, x a distance in the downstream direction. The correlation functions of bed elevation η and slope $\eta' = d\eta/dx$ are defined as follows, respectively,

$$K_{\eta\eta}(\xi) = \overline{\eta(x)\eta(x+\xi)}, \quad K_{\eta'\eta'}(\xi) = \overline{\eta'(x)\eta'(x+\xi)}. \tag{4}$$

They are related to each other by

$$K_{\eta'\eta'}(\xi) = -d^2K_{\eta\eta}(\xi)/d\xi^2. \tag{5}$$

The spectra of η and $d\eta/dx$ are expressed as two-sided Fourier transforms of $K_{\eta\eta}$ and $K_{\eta'\eta'}$, viz.

$$S_{\eta\eta}(k) = \int_{-\infty}^{\infty} K_{\eta\eta}(\xi)\, e^{-i2\pi k\xi}\, d\xi, \tag{6}$$

and

$$K_{\eta\eta}(\xi) = \int_{-\infty}^{\infty} S_{\eta\eta}(k)\, e^{i2\pi\xi k}\, dk, \tag{7}$$

where k is a wave-number in cyc/unit length, and similarly for $S_{\eta'\eta'}$.

In the spectral representation, (5) reduces to

$$S_{\eta\eta}(k) = \frac{S_{\eta'\eta'}(k)}{(2\pi k)^2}. \tag{8}$$

In the fully developed sand waves, the slope of sand beds can never exceed the angle of repose of sand particles, ϕ. In fact, in the well-developed sand beds, there are observed the wave crests of surface configuration, i.e. the discontinuities of surface gradient which in the spectral terms correspond to the existence of a certain form of the spectrum of large wave-numbers. As a consequence, there would exist an equilibrium range where the spectral form is governed predominantly by ϕ and the wave-number k.

For lower wave-number or frequency regions, on the other hand, the instability mechanism between sand beds and water flowing over them (Cartwright 1959; Kennedy 1963) prevails, thus the law of equilibrium subrange fails at low wave-number. The mechanisms of sand-wave generation are similar to those of wind waves in the points that the instability and resonance mechanism proposed by Miles (1957) and Phillips (1957) prevails for low wave-number regions and the equilibrium spectrum (Phillips 1958) exists for higher wave-numbers.

The function $S_{\eta'\eta'}(k)$ has the dimension of length and the angle of repose ϕ is non-dimensional. Thus, for the equilibrium subrange, it follows immediately on dimensional grounds that

$$S_{\eta'\eta'}(k) = (2\pi)^2\alpha(\phi)\,k^{-1} \quad (k_0 \ll k \ll d^{-1}), \tag{9}$$

where $\alpha(\phi)$ is a certain function of ϕ. However, since ϕ does not vary much, $\alpha(\phi)$ may be treated as a constant. d is a diameter of sand particle and k_0 represents the smallest wave-number of the wave field in which the interactional instability mechanisms are not important.

From (8) and (9), the sand-wave spectrum for the equilibrium range is derived as

$$S_{\eta\eta}(k) = \alpha(\phi)\,k^{-3} \quad (k_0 \ll k \ll d^{-1}). \tag{10}$$

Frequency spectrum

Almost the same discussion holds for the frequency spectra of the elevation η as well as of the rate of temporal change of bed configuration at a point of observation $\dot{\eta} = \partial\eta/\partial t$ which are symbolized by $P_{\eta\eta}(f)$ and $P_{\dot{\eta}\dot{\eta}}(f)$, respectively.

$P_{\eta\eta}(f)$ is related to $P_{\dot{\eta}\dot{\eta}}(f)$ as

$$P_{\eta\eta}(f) = P_{\dot{\eta}\dot{\eta}}(f)/(2\pi f)^2, \tag{11}$$

where f denotes the frequency in unit of cyc/unit time.

Now, the change of bed configuration continues as long as the bed material moves. The movement of sand bed is predominantly determined by the shear velocity U_* or the bottom shear stress τ_0. Factors such as the densities of water and sand particle (ρ_0 and ρ_s), the size of sand particle d and the gravitational acceleration g are also to be considered in the dimensional analysis. As well known in the theory of the critical tractive force, however, these factors will always appear as a non-dimensional group in the form of $\psi = U_*^2/([\rho_s/\rho_0]-1)gd$. Then, the frequency spectra of sand waves in the equilibrium subrange are given in the following forms,

$$P_{\dot{\eta}\dot{\eta}}(f) = (2\pi)^2 f_n(\psi)\,U_*{}^2 f^{-1} \quad (f_1 < f < f_\infty) \tag{12}$$

and

$$P_{\eta\eta}(f) = f_n(\psi)\,U_*{}^2 f^{-3} \quad (f_1 < f < f_\infty), \tag{13}$$

where f_1 and f_∞ mean the lower and the upper frequency boundaries of the equilibrium subrange in the frequency spectra, respectively, and $f_n(\psi)$ is a certain function of ψ.

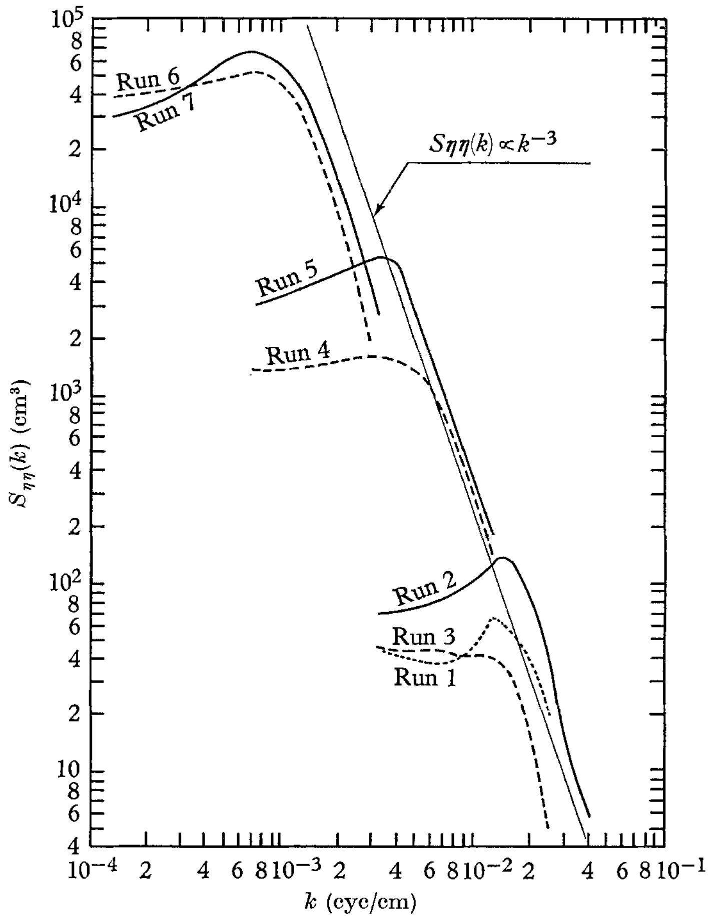

FIGURE 1. Wave-number spectra of sand waves compared with the '-3 power law, (10). (Curves are replotted from the graphs by Nordin & Algert (1966).)

The velocity of sand-wave movements $c(k)$ is generally related to k and f as

$$f = c(k)\,k. \tag{14}$$

On the other hand, $P_{\eta\eta}(f)$ and $S_{\eta\eta}(k)$ have a relation that

$$P_{\eta\eta}(f) = S_{\eta\eta}(k)\,(dk/df). \tag{15}$$

Therefore, in order that the two spectral forms (10) and (13) derived on the dimensional ground are consistent with each other, the velocity of sand-wave movement $c(k)$ needs to have a constant value irrespective of wave-number k,

$$c^2(k) = f_n(\psi)\,U_*{}^2/\alpha. \tag{16}$$

However, in the lower wave-number ranges, interactions between the sand waves and the water flowing over them become appreciable to consider $c(k)$ dependent on k (Cartwright 1959; Kennedy 1963),

$$c(k) \propto k \coth(2\pi kh), \tag{17}$$

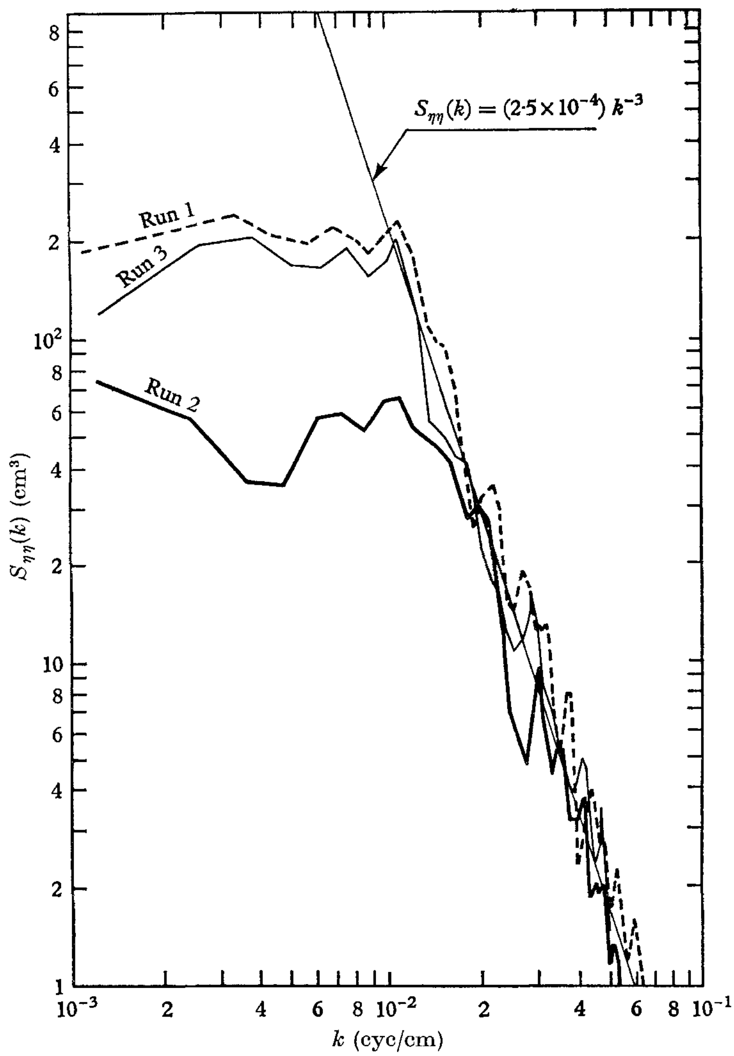

FIGURE 2. Wave-number spectra of sand waves by Ashida & Tanaka (1967) compared with the '-3 power law' prediction, (10). (The scales of ordinate of the original figures are reduced by a factor ten, because values of variances of elevation $\overline{\eta^2}$ obtained by the integration of the spectra are larger just by a factor ten than the values of $\sigma^2 = \overline{\eta^2}$ listed in table 2 of their original paper which are reasonable values from our experience.)

where h means the depth of flow. As a first approximation when k is large enough compared with $1/(2\pi h)$, $c(k)$ is represented by

$$c(k) = \gamma k, \tag{18}$$

where γ is a constant.

On the other hand, from the mechanism of sand-wave movements, it will be readily appreciated that the equilibrium range of spatial spectrum extends for far lower wave-numbers than for the equilibrium range of frequency spectrum. Then, from (10), (14), (15) and (18), the frequency spectrum $P_{\eta\eta}(f)$ for a relatively lower frequency region is described as

$$P_{\eta\eta}(f) = \tfrac{1}{2}\alpha(\phi)\gamma f^{-2} \quad (f_0 < f < f_1). \tag{19}$$

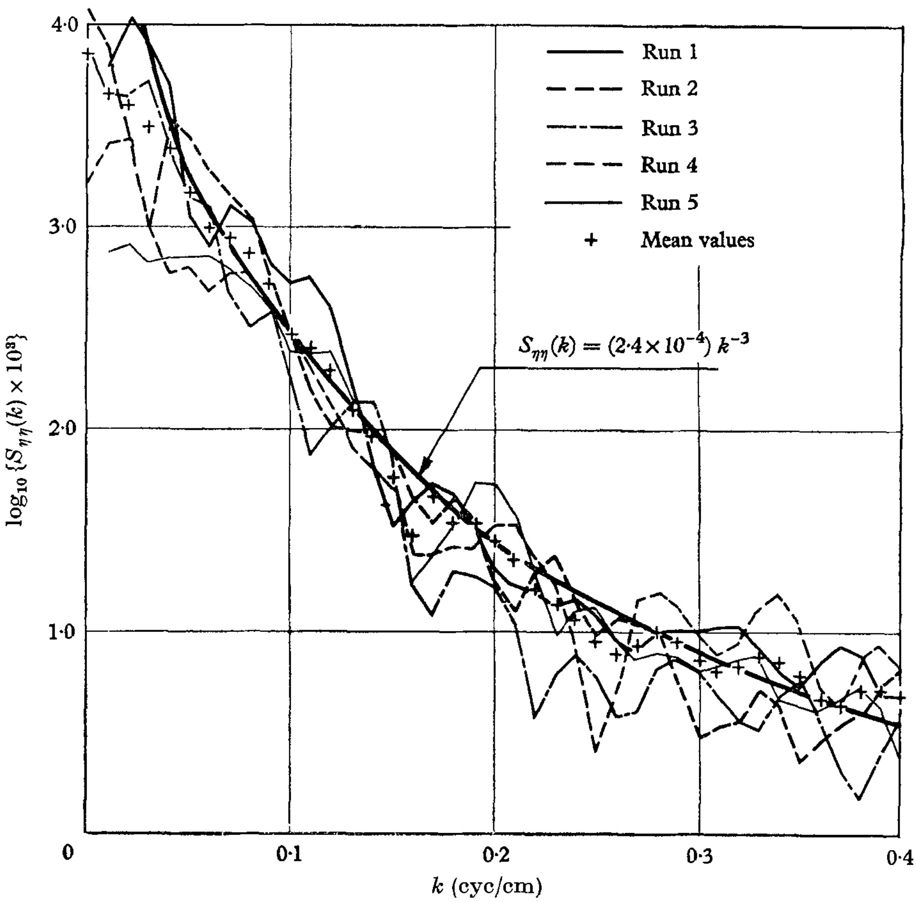

FIGURE 3. Wave-number spectra of sand waves by Fukuoka (1968) compared with the '−3 power law' prediction, (10). Crosses in the figure are mean values of five data and a solid curve represents the '−3 power law'.

3. Comparisons with experimental data

Recently some precise experimental data on the spectral properties of sand waves have been published by Nordin & Algert (1966), Ashida & Tanaka (1967) and Fukuoka (1968). These data are shown in figures 1 to 4 compared with the theory just derived above. The characteristic features of experimental conditions are given in table 1. Except runs 6 and 7 of Rio Grand Conveyance Channel by Nordin & Algert, all of them are obtained from laboratory experiments. Runs

Experimenters	Run	Width of channel	Depth of flow, h (cm)	Channel slope	Discharge rate (l/s)	U_* (cm/s)	Sand diameter	Types of sand waves (cyc/cm)	Critical wave-number k_0	$\alpha(\phi)$	Mean of $\alpha(\phi)$
Ashida &	A-1		11·64	1/400	30	4·37		Dune	0·012	$2{\cdot}0\times10^{-4}$	
Tanaka (1967)	A-2	50 cm	13·89	1/310	50	5·30	d_m = 0·732 mm	Dune	0·018	$2{\cdot}7\times10^{-4}$	$2{\cdot}5\times10^{-4}$
	A-3		7·31	1/600	15	3·03		Dune	0·012	$2{\cdot}8\times10^{-4}$	
Fukuoka	1		4·4	1/300	7·5	3·8		Dune	0·03	$2{\cdot}5\times10^{-4}$	
(1968)	2	40 cm	4·3	1/180	7	4·84	d_m = 0·18 mm	Dune	0·05	$2{\cdot}3\times10^{-4}$	
	3		3·6	1/150	7	4·85		Flat bed	0·03	$2{\cdot}0\times10^{-4}$	$2{\cdot}4\times10^{-4}$
	4	40 cm	2·8	1/400	5	2·52	—	Standing wave	0·07	$2{\cdot}7\times10^{-4}$	
	5		3·85	1/230	7	4·05	—	Dune	0·03	$2{\cdot}4\times10^{-4}$	
Nordin &	1	12·19 cm	12·2	0·0037	19·7	3·84		Dune	0·018	$3{\cdot}0\times10^{-4}$	
Algert (1966)	2		14·8	0·0038	24·5	4·00		Dune	0·020	$5{\cdot}0\times10^{-4}$	$3{\cdot}07\times10^{-4}$
	3		17·7	0·0044	29·7	4·41		Dune	0·015	$1{\cdot}2\times10^{-4}$	
	4	2·44 m	20·4	0·00136	36·3	4·82	Fine sand	Dune	0·0066	$3{\cdot}0\times10^{-4}$	$3{\cdot}25\times10^{-4}$
	5		32·0	0·00134	62·9	5·76		Dune	0·0039	$3{\cdot}5\times10^{-4}$	
	6	27·43 m	79·5	0·00058	266·2	6·53		Dune	0·0016	$0{\cdot}8\times10^{-4}$	$0{\cdot}9\times10^{-4}$
	7		127	0·00058	291·7	8·11		Dune	0·0013	$1{\cdot}0\times10^{-4}$	

TABLE 1. The proportionality constant of the '-3 power law' $\alpha(\phi)$ derived from several sources, together with characteristics of experiments.

4 and 5 by Fukuoka are performed under somewhat artificial conditions. At the upstream end of the channel, surface waves generated by a vibrating horizontal bar are imposed to form a considerably ruffled stream.

Although the experiments cover relatively wide hydraulic conditions of channel dimensions, sand diameters, channel slopes and bed configurations, the

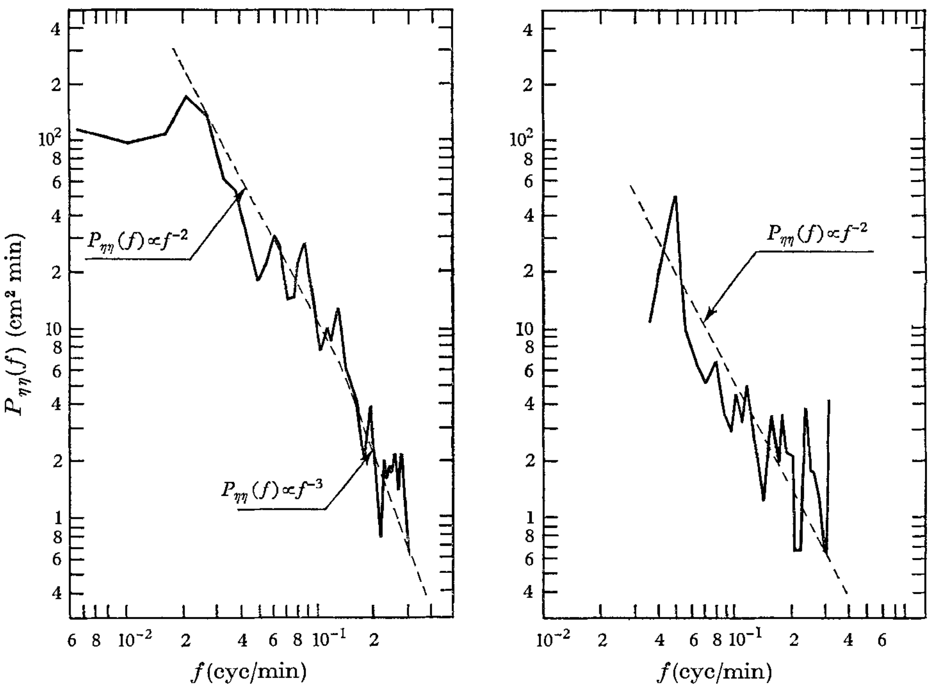

FIGURE 4. Frequency spectra of sand waves by Ashida & Tanaka compared with (13) and (19).

spectral form of the '-3 power law' exists for a wide range of high wave-numbers and frequencies. The lower limit of frequency of the power law depends quite strongly on the channel dimensions. From table 1 and figure 5 a rough estimation of k_0 is derived as

$$k_0 \simeq 0{\cdot}15/h, \tag{20}$$

or an estimation of the wavelength of a prevailing sand wave, L_0, is given by

$$L_0 \simeq 7h. \tag{21}$$

The proportionality constant $\alpha(\phi)$, being affected by the angle of repose of sand, scatters as given in table 1. The values of the constant determined from Japanese data coincide with each other in spite of the differences in the sand diameter and bed configurations. The mean value of these data, except runs 6 and 7 by Nordin and Algert, is $2{\cdot}8 \times 10^{-4}$, giving the following spectral function in the average

$$S_{\eta\eta}(k) = 2{\cdot}8 \times 10^{-4} k^{-3}. \tag{22}$$

Data on the frequency spectrum are too scarce to derive any conclusive results. However, the frequency spectra (figure 4) by Ashida & Tanaka give a good support to both the '-3 power law' range, (13), and the '-2 power law' range, (19). It is also shown that the spectrum of ripples (figure 4*b*) follows largely the '−2 power law'.

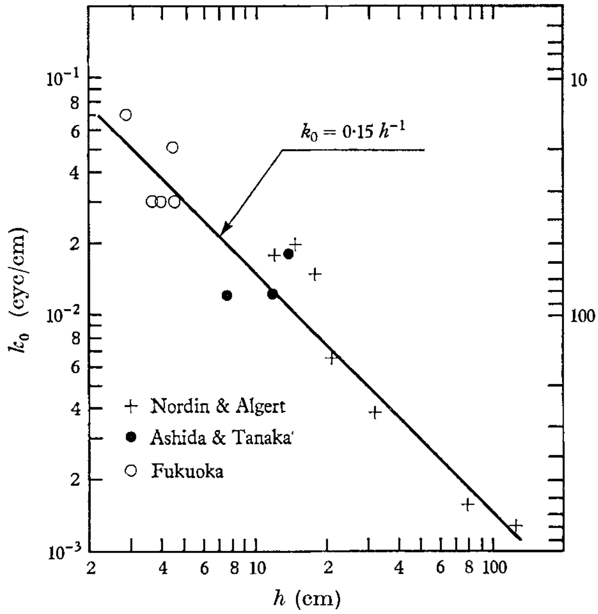

FIGURE 5. Relation between the lower limit wave-number k_0 of the '−3 power law' and the depth of flow h.

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